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Efficient Successive Reanalysis Technique for Engineering Structures

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Engineering structural design requires many simulations of the structural system after modifications are made to the initial structure. The computational cost of these repetitive simulations can be prohibitive in many engineering applications, such as optimization, reliability analysis, and so forth. In this work, the successive matrix inversion method is improved to include the capability to update not only the inverse of the modified stiffness matrix, but also the modified system response vector. This is done by introducing an influence vector storage matrix and a vector-updating operator that makes it possible to perform sequential reanalyses. The improved successive matrix inversion method has a wider applicable range of modification than the original method. The applicability of the proposed successive matrix inversion method is demonstrated with two popular engineering problems: sensitivity analysis and reliability analysis of an aircraft wing structure.

Nomenclature

$\{d_0\}$, $\{d\}$	=	original and modified displacement vectors
$\{f\}$	=	load vector
$[I]$	=	identity matrix
$[K_0]$, $[K]$	=	original and modified global stiffness matrices
m	=	total number of columns that have nonzero elements in $[\Delta K]$
m_i	=	m in ith sequential reanalysis stage
N	=	total number of degree-of-freedom
q	=	modification ratio (m/N)
α_i	=	ith scale factor of given elements' thicknesses
β_i	=	ith scale factor of elastic modulus for ith region in the target structure
$[\Delta K]$	=	stiffness modification matrix

Superscripts

j	=	successive stage number
T	=	transposition

Subscript

j	=	jth element of a vector

I. Introduction

ENGINEERING structural design requires many simulations of the structural system after modifications are made to the initial structure. This is especially true for successive design steps of structural systems, such as optimization, reliability analysis, and so forth, in which many repetitive simulations are performed. However, the total computational cost of the processes can be prohibitive. Therefore, many reanalysis techniques are developed in an effort to reduce the cost by making use of previous analysis results in subsequent system analyses.

There are two general categories of reanalysis techniques: surrogate methods and direct methods. General reviews of reanalysis methods can be found in some literatures [1,2]. Surrogate methods generally construct an approximation model of a specific response of a target system with minimum interactions with an original system analyzer (usually a "black box"), such as an intensive computer program of finite element analysis (FEA). The approximation model is usually constructed as a simple, closed-form equation based on series expansion [3–6] or design of experiments [7–9]. Surrogate methods have been extensively demonstrated in engineering disciplines and successively applied to many engineering designs, such as optimization, reliability analysis, optimization based on reliability, and so forth. However, the solutions of the surrogate methods are valid only within certain bounds. The valid bounds depend on the efficiency of the surrogate method and the characteristics of the original system. One of the more robust ways to increase the accuracy and efficiency of an applied surrogate method is to provide a more exact model of the original system. On the other hand, direct methods of structural system reanalysis techniques obtain the response of the modified structural system by using linear equations of the discretized system directly. Direct methods include iterative methods [10,11], the Sherman–Morrison and Woodbury (SMW) formulas [12,13], the successive matrix inversion (SMI) method [14], and so forth. Iterative methods are found to be effective only for small changes in a design and for a sparse stiffness matrix. Moreover, the iterative procedure should be continued until the solutions are converged, and the convergence rate might be slow or even divergent in certain numerical conditions of the stiffness matrix. Because the SMW formulas have been introduced, there have been many efforts to incorporate SMW in structural reanalysis [15]. The application of SMW is limited to modifications on either extremely small portions of an initial structure or a specific type of element (e.g., truss) in FEA.

In this paper, the SMI method [14], which has the equivalent formula of SMW, but independently derived from the binomial series expansion (BSE), is improved to include the capability to

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update both the inverse of the modified stiffness matrix and the modified response vector. It is found that the proposed SMI method is more efficient than the lower and upper triangular matrix (LU) decomposition method in solving not only a reanalysis problem, but also an initial and complete analysis problem. This improvement becomes possible by introducing an influence vector storage matrix and a vector-updating operator into SMI. The improved SMI method has a wider applicable range of modification than the original one [14] and the computational cost is significantly reduced. Updating processes for both the modified inverse matrix and the modified response vector can be used in a combined way for many sequential reanalyses to reduce the overall computational cost. The applicability of the proposed SMI method is demonstrated with two popular engineering problems: sensitivity analysis and reliability analysis of an aircraft wing structure.

II. Improved Successive Matrix Inversion Method

In the FEA procedure, most of the computational cost is incurred in inverting or decomposing the stiffness matrix of an engineering structural system to solve the equilibrium equations simultaneously. In a sequential analysis, the target structure is changed with small modifications. Hence, the main idea of reanalysis techniques is to regenerate the modified system response efficiently without another complete system analysis. First, we briefly review the previous SMI method.

A. Review of the SMI Method

The SMI method updates the inverse of the stiffness matrix by considering only the modified portion of the stiffness matrix for the reanalysis of the modified structure. Assume that the initial simulation given by Eq. (1) is performed using FEA.

$$[K_0]\{d_0\} = \{f\} \tag{1}$$

From this initial analysis, the inverse of the stiffness matrix $[K_0]^{-1}$ and the initial response vector $\{d_0\}$ are available. In a sequential analysis, the structural design is changed as follows:

$$([K_0] + [\Delta K])\{d\} = \{f\} \tag{2}$$

According to the previous SMI [14], the evaluation of the modified response vector required premultiplying Eq. (2) by $[K_0]^{-1}$ to give

$$([I] + [K_0]^{-1}[\Delta K])\{d\} = [K_0]^{-1}\{f\} \tag{3}$$

We assume for convenience that the first m columns of $[\Delta K]$ have nonzero elements. For Eq. (3), the BSE is considered to obtain the inverse of $[I] + [K_0]^{-1}[\Delta K]$. The infinite series expansion terms are explicitly obtained by decomposing the stiffness modification matrix into column vectors as shown in Eq. (5). As a result, the inverse of the modified stiffness matrix is obtained by a successive inversion procedure with decomposed column vectors of $[\Delta K]$ using the following three equations:

$$\{B^{(j)}\} = -[K^{(j-1)}]^{-1}\{\Delta K^{(j)}\} \tag{4}$$

$$\{Bs^{(j)}\} = \{B^{(j)}\}/(1 - \{B^{(j)}\}_j) \tag{5}$$

$$[K^{(j)}]^{-1} = [K^{(j-1)}]^{-1} + \{Bs^{(j)}\}\{Kb^{(j-1)}\}^T \tag{6}$$

Furthermore, $\{\Delta K^{(j)}\}$ is the jth column vector of $[\Delta K]$, $\{Kb^{(j-1)}\}$ is the jth row vector of $[K^{(j-1)}]^{-1}$, and the initial $[K^{(0)}]^{-1}$ is given as $[K_0]^{-1}$. The required number of successive steps is the number of nonzero columns in $[\Delta K]$. Because the inverse of the modified stiffness matrix, $([K_0] + [\Delta K])^{-1}$, is obtained in SMI, for the next modification, $[\Delta K_2]$, the inverse of the second modified stiffness matrix, $([K_0] + [\Delta K] + [\Delta K_2])^{-1}$, can be obtained by setting $([K_0] + [\Delta K])^{-1}$ as a new initial inverse of the stiffness matrix. For references to the SMI method, the reader is encouraged to read Bae et al [14]. Even though the SMI method was independently derived from BSE, the equations of SMI method are equivalent to those of the SMW formula. The SMW formula for a rank-one modification that can be expressed by two vectors as $[\Delta K] = \{u\}\{v\}^T$ is written as

$$\begin{aligned}([K_0] + \{u\}\{v\}^T)^{-1} &= [K_0]^{-1} \\ &- [K_0]^{-1}\{u\}(1 + \{v\}^T[K_0]^{-1}\{u\})^{-1}\{v\}^T[K_0]^{-1}\end{aligned} \tag{7}$$

With $[\Delta K]$, we can select $u = \{\Delta K^{(j)}\}$, the change in the column, and $\{v\} = \{e_j\}$, a unit vector with one in the jth position and zeroes elsewhere. By obtaining $r^{(j)} = \{v\}^T[K_0]^{-1}\{u\}$, the SMW formula becomes

$$([K_0] + [\Delta K]^{(j)})^{-1} = [K_0]^{-1} - (1 - r^{(j)})^{-1}[K_0]^{-1}[\Delta K]^{(j)}[K_0]^{-1} \tag{8}$$

which is essentially the same with SMI, shown in Eqs. (4–6). However, the following section introduces a new reanalysis formulation of SMI with a new vector-updating operator to enhance the performance of SMI in engineering applications.

B. Improved SMI method

For Eq. (3), we formulate another problem whose initial matrix is $[I]$. The modification matrix is $[B] = -[K_0]^{-1}[\Delta K]$ and the right side of Eq. (3) is the initial response $\{d_0\}$. The ultimate purpose of this formulation is to obtain the influence matrix, $[S] = ([I] + [B])^{-1}$, which updates the initial response to the modified response with respect to the given modification matrix as

$$\{d\} = [S]\{d_0\} \tag{9}$$

As in the SMI procedure, by decomposing $[B]$ into column vectors, the influence matrix is updated from the initial identity matrix by successive procedures with the following three equations:

$$\{B^{(j)}\} = [S^{(j-1)}]\{B^{(j)}\} \tag{10}$$

$$\{Bs^{(j)}\} = \{B^{(j)}\}/(1 - \{B^{(j)}\}_j) \tag{11}$$

$$[S^{(j)}] = [S^{(j-1)}] + \{Bs^{(j)}\}\{Sb^{(j-1)}\}^T \tag{12}$$

where $\{Sb^{(j-1)}\}$ is the jth row vector of $[S^{(j-1)}]$. Compared with the previous SMI method, this procedure is more cost-effective because only the influence matrix, which is initially $[I]$, is updated successively, rather than the whole inverse of the stiffness matrix. The column vector of $[B]$ is required to be altered by the influence matrix, as shown in Eq. (10) at each step. Note that the modified stiffness matrix, $([K_0] + [\Delta K])^{-1}$, can be obtained as $[S][K_0]^{-1}$. For the next modification, $[\Delta K_2]$, the inverse of the second modified stiffness matrix, $([K_0] + [\Delta K] + [\Delta K_2])^{-1}$, can be computed as $[S_2][S][K_0]^{-1}$ sequentially. Hence, not only the modified response vector, but also the inverse of the modified stiffness matrix, can be tracked through the influence matrix. This means that the influence matrix makes it possible to perform sequential reanalyses.

However, close examination of the equations reveals that in Eq. (12), the updated $[S]$ matrix, which is started with an identity matrix, is also unnecessary. The jth column of $[S^{(j-1)}]$ is filled up with $\{Bs^{(j)}\}$ and the previously updated $(j-1)$ columns in $[S^{(j-1)}]$ are updated due to the jth column, $\{Bs^{(j)}\}$. Because of this updated influence matrix $[S^{(j)}]$, at the next step the column vector of $[B]$ is directly changed by matrix-vector multiplication as shown in Eq. (10), which is a kind of simultaneous superposition operation for the jth column of $[B]$. However, if a successive vector-updating scheme is employed instead of Eqs. (10) and (12), the process of updating the influence matrix, Eq. (12), can be avoided. Because the successive scheme is updating only vectors, no additional cost is required beyond the computation cost of Eq. (10). Hence, to save the unnecessary computational cost in the SMI procedure, updating the influence matrix, Eq. (12), is skipped, and a new influence vector storage (IVS) matrix and a new vector-updating operator are introduced.

The IVS matrix $[P]$ which eventually becomes an $N \times m$ matrix for m columns modification to an initial stiffness matrix, starts with

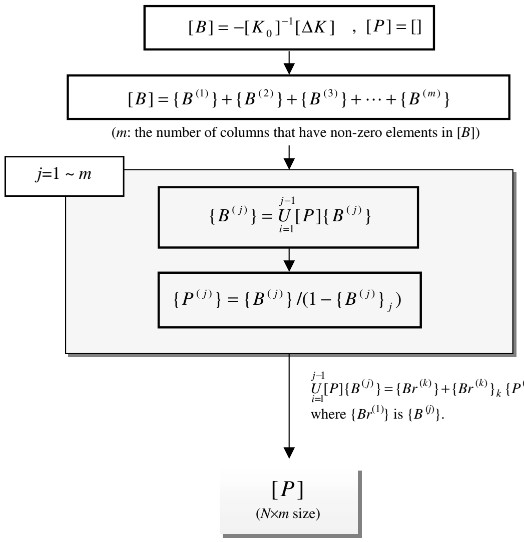


Fig. 1 Improved successive matrix inversion algorithm for the *m* columns modification.

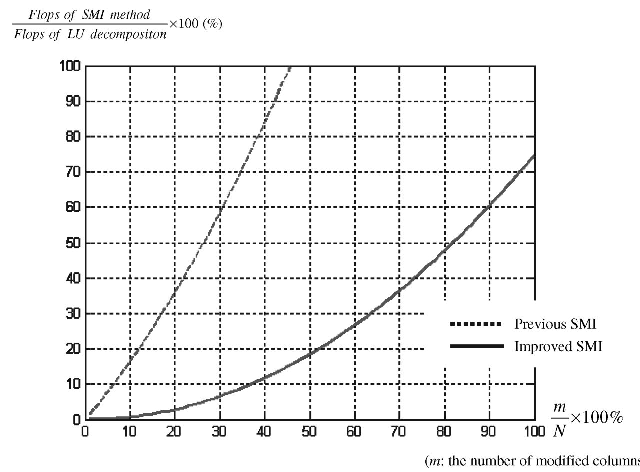


Fig. 2 Compared cost ratios of SMI to LU decomposition.

an empty zero-order matrix. At the first updating step, the first column vector of $[B]$ is not changed because $[P]$ is empty, and the manipulated vector $\{Bs^{(1)}\}$ is stored in the first column of $[P]$. At the next stage, the second column vector of $[B]$ is updated by the IVS matrix $[P]$ as follows:

$$\{B^{(2)}\} = \{B^{(2)}\} + \{P^{(1)}\} \times \{B^{(2)}\}_2 \tag{13}$$

The vector, $\{Bs^{(2)}\} = \{B^{(2)}\}/(1 - \{B^{(2)}\}_2)$, is stored at the second column of $[P]$. At the jth stage, the IVS matrix becomes an $N \times (j-1)$ matrix, and the jth column vector of $[B]$ is updated sequentially with the $(j-1)$ columns of $[P]$ one by one as follows:

$$\{Br^{(k+1)}\} = \{Br^{(k)}\} + \{Br^{(k)}\}_k\{P^{(k)}\} \qquad k = 1 \cdots j-1 \tag{14}$$

where $\{Br^1\}$ is $\{B^{(j)}\}$, and $\{Br^{(j)}\}$ is the updated vector $\{B^{(j)}\}$, which is the same as the vector from Eq. (10). For convenience, the operation of Eq. (14) is expressed from now on with a new successive vector-updating operator U as follows:

$$\{B^{(j)}\} = U_{i=1}^{j-1}[P]\{B^{(j)}\} \tag{15}$$

After the computations for the $\{B^{(j)}\}$ vector in Eq. (15), $\{B^{(j)}\}$ is simply stored at the jth column of $[P]$. In summary, at the jth stage, $\{B^{(j)}\}$ is sequentially updated as shown in Eq. (15), instead of by the simultaneous superposition operation shown in Eq. (10). Then, the IVS matrix $[P]$ simply stores the $\{B^{(j)}\}$ vector in its corresponding column and becomes a matrix with $N \times j$ size, without the updating procedure as in Eq. (12). The procedure for the proposed SMI method is shown in Fig. 1. Finally, after obtaining $[P]$ ($N \times m$ size) for all nonzero columns of $[B]$, the modified response vector is obtained as follows:

$$\{d\} = U_{i=1}^{m}[P]\{d_0\} \tag{16}$$

When the nonzero columns of $[\Delta K]$ are scattered randomly, one vector that has the information of the locations of nonzero columns in $[\Delta K]$ might be needed and considered in the SMI procedures.

III. Some Computational Issues of Using SMI

The computational cost of SMI, expressed by the number of the floating point operations (flops), is compared with that of a popular direct method, LU decomposition. One flop is approximately the work required to compute one addition and one multiplication. Because the SMI method gives an exact solution for the symmetric and nonsymmetric modification matrices, the LU decomposition method (instead of the Cholesky decomposition method) is selected as a direct complete analysis method to compare the efficiency with SMI. For an $N \times N$ matrix, the LU decomposition method requires $2/3N^3$ flops to solve the system, which is modified with any rank size. However, by using the proposed SMI method, the computational cost depends on the size of modified rank from the initial stiffness matrix. Obviously, from Eq. (15), the cost of the proposed SMI method is about $\frac{1}{2}(m-1)mN$ for m columns modification in the stiffness matrix. Figure 2 shows the ratio of the computational cost of SMI to LU decomposition. It is found from Fig. 2 that the reanalysis cost for a 50% rank modification to the initial stiffness matrix is less than 20% of the complete analysis cost using LU decomposition. It is noted that even for a full modification of the $N \times N$ stiffness matrix, the SMI method is more efficient than the conventional LU decomposition method, with about a 25% cost savings. Moreover, there is no pivoting procedure in SMI.

In many practical engineering structural problems, computational methods such as FEA lead to a sparse matrix that has a small number of nonzero elements. Even though the stiffness matrix is very sparse, it is very hard to take advantage of the sparseness in most direct solvers using decomposition methods because of the unpredicted fill-in in the process of decomposition. However, in the proposed SMI method, the sparseness of the stiffness matrix can be considered more explicitly to obtain the computational benefit. Because the modification stiffness matrices in engineering structural systems are usually very sparse, the cost of obtaining $[B]$ is ignored throughout this paper. For a specially structured matrix such as a diagonally banded stiffness matrix, the IVS matrix $[P]$ can be obtained by efficient systematic computations due to the pattern of sparseness in $[K]$.

In a specific analysis, such as reliability analysis, design optimization, and so on, many simulations may be required for sequential modifications to a target structure. By using SMI, the sequential reanalysis can by performed by accumulating the influence vectors sequentially in $[P]$. For example, the first modified response $\{d_1\}$ is obtained with the initial response $\{d_0\}$ and $[P]$ for the m_1 rank modification as follows:

$$\{d\} = U_{i=1}^{m_1}[P]\{d_0\} \tag{17}$$

For the next modification, $[\Delta K_2]$ with m_2 nonzero columns, the additional influence vectors are accumulated in the previous $[P]$ matrix. The second modified response $\{d_2\}$ can be computed as follows:

$$\{d_2\} = U_{i=m_1+1}^{m_1+m_2}[P]\{d_1\} = U_{i=1}^{m_1+m_2}[P]\{d_0\} \tag{18}$$

Hence, for the kth sequentially modified system with m_k modified columns, only the additional m_k times updating processes in Eq. (15) are required with the kth initial $[P]$ matrix, the size of which becomes $N \times (m_1 + m_2 + \ldots m_{k-1})$. However, as the size of the column of $[P]$ increases in sequential modifications, the cost for the updating process in Eq. (15) increases exponentially. Therefore, in the case of many sequential reanalyses with small change fractions, the intermediate process of updating the inverse of the modified stiffness matrix can reduce the overall computational cost by decreasing the sequential updating processes in Eq. (15). For example, when T_n sequential reanalyses are required with q modification ratio to N, that is, $m_k(=q \times N)$ independent modified columns relative to the previous columns, the total cost (flops) of T_n sequential reanalyses is as follows:

$$\mathrm{SSMI}_{\mathrm{cost}} = \sum_{j=1}^{T_n} \left(\sum_{i_1=1}^{qN-1} \sum_{i_2=1}^{i_1} N + (j-1)qN^2N \right) \tag{19}$$

In the right side of the preceding equation, the first term is for finding $[P]$ with qN modified columns and the second term is for the additional cost of updating $\{B\}$ in Eq. (15) with the previously stored updating vectors in $[P]$. On the other hand, if the inverse stiffness is updated d_n times in T_n sequential reanalyses, the total cost (flops) is computed as follows:

$$D = \frac{K}{d_n + 1} \tag{20}$$

$$\mathrm{pSMI} = \sum_{j=1}^{D} \left(\sum_{i_1=1}^{qN-1} \sum_{i_2=1}^{i_1} N + (j-1)qN^2N \right) \times (d_n + 1) \tag{21}$$

$$\mathrm{UK} = DqN^3 d_n \tag{22}$$

$$\mathrm{TSMI}_{\mathrm{cost}} = \mathrm{pSMI} + \mathrm{UK} \tag{23}$$

where pSMI is the cost of $d_n + 1$ SMI procedures with D sequential reanalyses at each procedure, and UK is the cost of updating the inverse of stiffness matrix d_n times. The marginal number of sequential reanalyses, T_m, for $\mathrm{SSMI}_{\mathrm{cost}}$ against $\mathrm{TSMI}_{\mathrm{cost}}$ can be found by solving the following problem:

$$T_m = \left\{ T_n \middle| \frac{\mathrm{SSMI}_{\mathrm{cost}} - \mathrm{TSMI}_{\mathrm{cost}}}{\text{LU decomposition cost}} = \frac{3d_n T_n q(qT_n - 2)}{4(1 + T_n)} > 0 \right\} \tag{24}$$

It is obvious that the marginal T_m is $2/q$ from Eq. (24). The marginal T_m indicates that it is better to employ the process of updating the stiffness matrix for more than T_m sequential reanalyses in a sense of overall cost savings. For sequential reanalyses more than T_m, the minimum $\mathrm{TSMI}_{\mathrm{cost}}$ can be obtained by a sufficient number of d_n as follows.

$$\text{Minimum } \mathrm{TSMI}_{\mathrm{cost}} = \lim_{d_n \to T_n \& N \to \infty} \frac{\mathrm{TSMI}_{\mathrm{cost}}}{\text{LU decomposition cost}} = \frac{3qT_n^2(2+q)}{4(1+T_n)} \tag{25}$$

For example, when $q = 0.01$, 1,000,000 simulation results can be obtained with about the cost of only 15,000 complete analyses by using SMI, that is, 1.5% of the cost of a complete solver using LU decomposition.

As mentioned earlier, the proposed SMI method fundamentally has the equivalent formulation to the Sherman–Morrison and Woodbury (SMW) formula even though the derivation and solution scheme or implementations are different. Hence it would be worthwhile to address some computational issues between SMW and SMI. As shown in Eq. (7), SMW needs to perform its update process considering the inverse of the stiffness matrix for any sequential modification in design iterations. Some techniques [16,17] using the SMW formulas have been developed to increase the computational efficiency by computing the modified displacement vector directly. In this case where only the displacement vector is updated, sequential reanalyses for serial modifications of different parts of a structure, which are the main processes of optimization and reliability analysis, cannot be performed successively. For these difficulties, SMI can perform the updating process more efficiently by introducing the influence vector storage matrix and a vector-updating operator as introduced in the previous section.

In updating the modified displacement vector, it is also pointed out by Akgün et al [15] that when the element of a truss structure is changed, SMW is efficient by using a rank-one modification property in $[\Delta K]$ of the truss element by considering an element by element modification rather than a column-by-column modification. However, if we take advantage of the same special property (the rank-one property) of $[\Delta K]$, the constant recursive terms in SMI can also be obtained very easily element by element as a special case of the SMI applications, without considering each column of $[\Delta K]$. A short description of using the rank-one property in SMI is presented as follows.

To find the recursive term in SMI of the modified displacement vector, let us define the $\{s_1\}$ vector, which is the first BSE term with the initial solution vector $\{d_0\}$, as follows:

$$\{s_1\} = [B]\{d_0\} = -[K_0]^{-1}[dK]\{d_0\} \tag{26}$$

The second BSE term is obtained by $\{s_1\}$ as follows:

$$\{s_2\} = [B]^2\{d_0\} = [B]\{s_1\} = -[K_0]^{-1}[dK]\{s_1\} \tag{27}$$

The recursive terms for each element of the s vector can be obtained by dividing each element of $\{s_2\}$ by that of $\{s_1\}$. It is noted that because of the rank-one modification, the recursive terms for all elements of the s vector are "the same constant." That is, after obtaining $\{s_1\}$, only one element of $\{s_2\}$ is sufficient to find the single-constant recursive term r.

$$r = \frac{\{s_2\}_1}{\{s_1\}_1} \tag{28}$$

As a result, the following simple equations are obtained.

$$\{d\} = \{d_0\} + \{s_1\}\frac{1}{1-r} \tag{29}$$

$$\{\Delta d\} = \{s_1\}\frac{1}{1-r} \tag{30}$$

For the computational cost of preceding operations, the major operation is only for obtaining the s_1 vector, which is trivial. As a result, by using the concept of the constant recursive term in BSE, SMI can update the displacement vector or the IVS matrix efficiently. As a brief summary, the comparison between SMW and SMI is presented in Table 1.

Table 1 Brief comparison between SMW and SMI

	SMW	SMI
Basic equations for a column-by-column update scheme	The original SMI gives the same equations of the variation of SMW	
Truss element modification (rank-one modification)	Efficient with decomposed vectors (uv^T)	More efficient than SMW by finding only a constant recursive term in BSE
Multiple general FE elements modification (multiple-rank modification)	SMI is more efficient than SMW by using the new procedure and new tools (IVS and SVU)	
Successive reanalysis scheme for serial structural modifications	Requires additional cost for performing the same procedure consecutively	Can be efficiently handled by updating only the IVS matrix

The SMI method can be implemented intrusively in current commercial FEA software packages as an additional/alternative solver for a system reanalysis technique with only minor modifications to the preexisting codes. Otherwise, the proposed method can be used as an external solver by accessing the initial global coefficient matrix, the force vector, and modified element stiffness matrices in a database of a FEA package. It is very feasible because many of current commercial FEA packages are providing the way for accessing their database.

IV. Numerical Examples

In this section, two examples of an intermediate complexity wing (ICW) structure are presented to demonstrate the efficiency of the proposed SMI method in engineering applications. The metallic structural model of ICW shown in Fig. 3 is a representative wing-box structure for a fighter aircraft. There are 62 quadrilateral membrane elements for the upper and lower skins and 55 shear elements for eight ribs and three spars. In the structural model, the wing has a root chord of 48 in, tip chord of 29.33 in, and a semi span of 90 in with a sweep angle of 26.8 deg.

A. The Application of SMI to Sensitivity Analysis

Sensitivity analysis is performed to find out the effect of design variables on a specific system response in a sequential design process such as design optimization. The direct differentiation method (DDM) is preferred to the finite difference method (FDM), because the design sensitivity for each design variable requires an additional finite element analysis in FDM. The sensitivity from FDM is also not exact due to truncation, round-off, and condition errors. However, FDM makes it possible to obtain sensitivity information with respect to design variables that may be hard to incorporate explicitly in the direct differential formulation. Hence, the proposed SMI method is employed to alleviate the major difficulty of FDM, the high computational cost. In this example, the linked design variables (a total of eight variables for the upper skins) in the ICW skin thicknesses are defined, as shown in Fig. 4. The design variable α_i in each region determines the actual skin thicknesses of the elements in the region as a scale factor to the initial values. The sensitivity information of the tip displacement for the design variable can be obtained by giving a small perturbation to each design variable and by checking the effect on tip displacement in FDM. Each design variable makes about a 10% modification to the initial stiffness matrix (the number of degrees of freedom, $N = 234$). The cost of SMI for one additional reanalysis for FDM is as low as 0.7% of that of the LU decomposition method, as seen in Fig. 2. Obviously, the total additional cost of sensitivity analysis using FDM with SMI for eight variables is only 5.6% of the one-time LU decomposition cost. The sensitivity information obtained by making a 0.01% perturbation at each design variable in FDM using SMI is shown in Fig. 5.

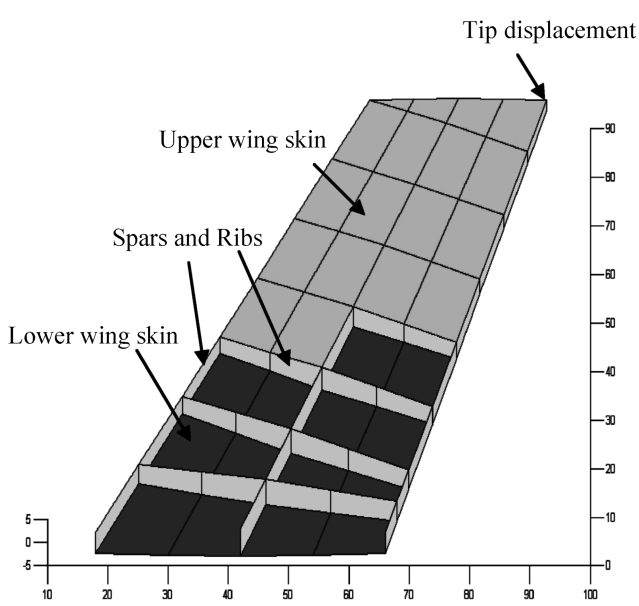


Fig. 3 Intermediate complexity wing structure model.

B. The Application of SMI to Reliability Analysis Using a Sampling Technique

Structural reliability analysis is performed to determine the probability of failure of a structure with a limit-state function in which a required performance of a target structure is defined. The limit-state function G separates the design space into failure and safe regions.

$$G(X) > 0, \qquad x_i \in \text{Failure region} \tag{31}$$

$$G(X) = 0, \qquad x_i \in \text{Failure boundary surface} \tag{32}$$

$$G(X) < 0, \qquad x_i \in \text{Safe region} \tag{33}$$

where $X(\in \Re^n)$ is a vector of uncertain parameters in the structural design including random loads, uncertain geometric dimensions, material properties, and so on. Each uncertain parameter is assumed to have an independent probability density function (PDF). With the limit-state function, the probability of failure P_f is computed as

$$P_f = \int_{G(X)<0} p(X)\,\mathrm{d}X \tag{34}$$

where $p(X)$ is the joint probability density function of X. In engineering structural reliability applications, numerical methods, such as the Monte Carlo simulation (MCS), can generally be performed to evaluate the multiple integration in Eq. (34). The crude MCS can be expressed as follows:

$$\hat{P}_f = \frac{1}{n}\sum_{i=1}^{n} I[G(X_i) > 0] \tag{35}$$

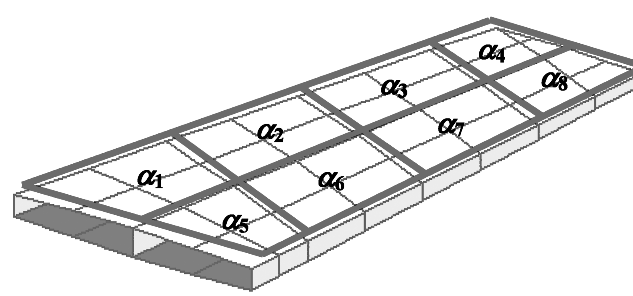
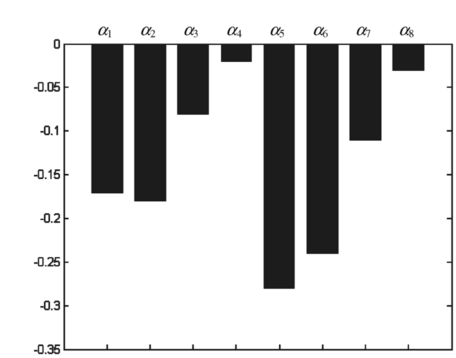


Fig. 4 Linked design variables (α_i) for ICW skin thickness.



	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
Sensitivity for $Disp_{tip}$	-0.17	-0.18	-0.08	-0.02	-0.28	-0.24	-0.11	-0.03

Fig. 5 ICW tip displacement sensitivities with skin thicknesses.

where X_i indicates a realization of random parameters from given PDFs, $\hat{P}_f$ represents the crude MCS estimator of failure probability, and n is the total number of MCS.

In this example, the proposed SMI method is applied to MCS to demonstrate its applicability to sequential repetitive reanalyses in structural reliability analysis. In Fig. 6, five scale factors (β_1–β_5) of an elastic modulus ($E = 1.05 \times 10^7$ psi) at different local parts of ICW are defined as random variables to describe a locally damaged situation. In MCS, samples are obtained from the Cartesian product of the samples of each random variable generated from each PDF. To describe the sequential procedure of MCS, a simple case that has only two random variables (β_1 and β_2) is shown in Fig. 7 as an example. For the first random variable (β_1), first stage reanalyses are performed from the initial design for the selected n_1 samples from the PDF of β_1, as shown in Fig. 7a. Then, as shown in Fig. 7b, for the n_2 samples of the second variable (β_2) from the given PDF, second stage reanalyses are performed by considering $[P]$, which is obtained from first stage reanalysis of the first random variable. As shown in Fig. 7a, the total computational cost of the first stage is only the number of samples (n_1) times the cost of SMI for the β_1 modification, which is about 0.68% of the complete analysis cost. The total cost of the second stage for β_2 modifications is the total number of second stage samples ($n_1 \times n_2$) times the cost for sequential SMI, which can be obtained from Eq. (21). The total cost is about 2.8% of the complete analysis cost. This means that if both n_1 and n_2 are 100, the results of 10,000 simulations are obtained while incurring only the cost of less than three complete analyses through SMI. For the current example with five random variables, SMI is applied in the same sequential way as in the case of two random variables. In this ICW example, the limit-state function is as follows:

$$G(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) = \frac{\mathrm{Disp_{tip}}(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)}{11.5\ (\mathrm{in})} - 1 \qquad (36)$$

Where

$$\beta_1 = \mathrm{Normal}[0.9, 0.1]$$

$$\beta_2 = \mathrm{Uniform}[0.7, 1.0]$$

$$\beta_3 = \mathrm{Uniform}[0.8, 1.0]$$

$$\beta_4 = \mathrm{Normal}[0.8, 0.1]$$

$$\beta_5 = \mathrm{Normal}[0.7, 0.1]$$

To obtain the failure probability for the displacement response of ICW, MCS is performed with 25 samples of each random variable and, as a result, gives about 0.42% failure probability. In this MCS, the total number of simulations is about 10,000,000. However, through the sequential reanalyses using the proposed SMI method, the computational cost of MCS is reduced to about 6.5% of the cost of using complete analyses of 10,000,000 simulations without reducing the number of total samples. Moreover, the successive SMI analyses for the successive random variable can be assigned to separate computers for an efficient parallel computation scheme.

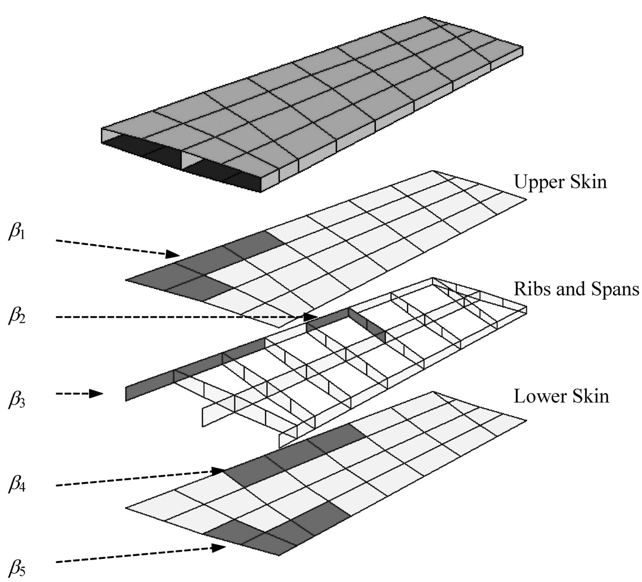
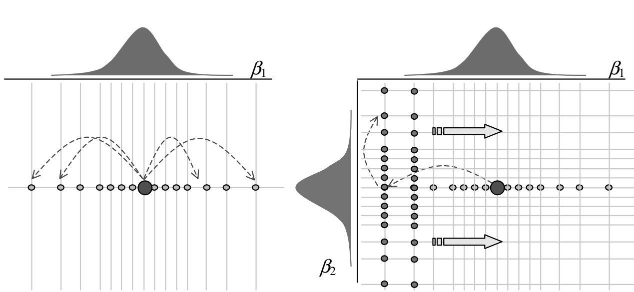


Fig. 6 Design variables (β_i) of elements that are subject to the elastic modulus uncertainty.



a) First stage reanalysis **b) Second stage reanalysis**

Fig. 7 Sequential computation for Monte Carlo simulation.

V. Summary

In this work, the SMI method is improved with respect to computational cost by introducing an updating vector storage matrix and a vector-updating operator. By comparing the computational cost (the number of flops), it is found that the proposed SMI method is even more efficient than the popular direct method, LU decomposition (by 25% less cost), which usually requires an additional pivoting procedure. For many sequential stages, the marginal number of stages is identified by Eq. (24), and it is found that the procedure of updating the inverse of the modified stiffness matrix is beneficial for many sequential stages more than the marginal number T_m. For computations of sensitivity, even though FDM can handle various design variables that are difficult to incorporate in a direct differential formulation, DDM is preferred to FDM due to the computational aspect. In this work, it is found that the SMI method can be used effectively for the FDM procedure by reducing the high computational cost of FDM significantly. By introducing the updating vector storage matrix and the vector-updating operator, sequential reanalysis can be performed in an explicit way and it is possible to handle many repetitive simulations, such as in the reliability analysis with sampling techniques, by using a parallel computational technique to enhance the computational speed. For the potential of the proposed SMI method, it is believed that, although SMI (which is efficient for localized modifications) is applied to obtain the exact solutions in this work, an efficient approximation scheme for a global modification can be developed using a combined procedure with SMI.

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